Probability Theory, Ph.D Qualifying, Fall 2016

Completely solve any five problems.

1. (a) Suppose that X and Y are independent random variables with the same exponential density
$$f(x) = \theta e^{-\theta x}, \ x > 0.$$
Show that the sum $X + Y$ and the ratio X/Y are independent.
(b) Show that the mean μ of a random variable X has the property
$$\min_c E(X - c)^2 = E(X - \mu)^2 = Var(X).$$

2. Show that for any two random variables X and Y with $Var(X) < \infty$,
$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)].$$

3. Show that random variables $X_n, \ n \geq 1$, and X satisfy $X_n \to X$ in distribution iff
$$E[F(X_n)] \to E[F(X)]$$
for every continuous distribution function F.

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution with $E(X_i) = 0$ and $Var(X_i) = 1$. Show that as $n \to \infty$,
$$Y_n = \frac{\sqrt{n} S_n}{\sum_{i=1}^n X_i^2} \to N(0, 1),$$
and
$$Z_n = \frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} \to N(0, 1),$$
where $S_n = X_1 + \cdots + X_n$.

5. Prove for iid random variables $\{X_n\}$ with $S_n = X_1 + \cdots + X_n$ that
$$\frac{S_n - C_n}{n} \to 0 \text{ a.s.}$$
for some sequence of constants C_n if and only if $E|X_1| < \infty$.

6. Let $\{X_n\}$ be iid random variables with $E|X_1| < \infty$. Show that $\sum (-1)^n \frac{X_n}{n}$ converges a.s.

7. If $\{X_n\}$ are iid $\mathcal{L}^1$ random variables, then $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converges a.s. if either (i) X_1 is symmetric or (ii) $E|X_1| \log^+ |X_1| < \infty$ and $EX_1 = 0$.

8. Let $\{\xi_n, \ n \geq 1\}$ be independent random variables such that for some $0 < p < 1$, $P(\xi_n = 1) = p$, $P(\xi_n = -1) = 1 - p = q$. For $n \geq 1$, let $\eta_n = \sum_{k=1}^n \xi_k$ and $\zeta_n = (q/p)^{\eta_n}$. Show that $\{\zeta_n, \ n \geq 1\}$ is a martingale.